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Outpaths in semicomplete multipartite digraphs

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Abstract

An outpath of a vertex x (an arc xy, respectively) in a digraph is a directed path starting at x (xy, respectively) such that x dominates the endvertex of the path only if the endvertex also dominates x. Firstly, we show that if D is a strongly connected semicomplete n-partite ($n \geqslant 3$) digraph, then every vertex v of D has an outpath of length $k-1$ for all $k \in \{3, 4, \ldots, n\}$. Our result generalizes a theorem of Moon (Canad. Math. Bull. 9 (1966) 297–301) for tournaments. Secondly, we show that if T is a regular n-partite ($n \geqslant 3$) tournament, then every arc of T has an outpath of length $k-1$ for all k satisfying $3 \leqslant k \leqslant n$. This result extends a theorem of Alspach (Canad. Math. Bull. 10 (1967) 283–286) for regular tournaments to regular multipartite tournaments.

Keywords: Cycle; Path; Regularity; Semicomplete multipartite digraph

1. Introduction

We consider finite digraphs without loops and multiple arcs. The vertex set of a digraph D is denoted by $V(D)$. If xy is an arc of a digraph D, then we say that x *dominates* y. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A dominates B and denote it by $A \to B$. The *outset* $N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *inset* $N^-(x)$ is the set of vertices dominating x. A digraph D is said to be *regular* if there is an integer r such that $|N^+(x)| = |N^-(x)| = r$ holds for every $x \in V(D)$.

Cycles and paths are always simple and directed. A k-cycle is a cycle of length k. A path of length p is called a *p-path*. A digraph D is *strongly connected* (or just *strong*), if for every ordered pair of distinct vertices $\{x, y\}$ there exists a path from x to y in D.

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A digraph obtained by replacing each edge of a complete n-partite graph with an arc or a pair of mutually opposite arcs is called a *semicomplete n-partite digraph* or a *semicomplete multipartite digraph*. A *multipartite tournament* is a semicomplete multipartite digraph without a 2-cycle and a tournament is an n-partite tournament having exactly n vertices.

Tournaments have rich structure. For example, Moon [7] proved that if T is a strong tournament on n vertices, then every vertex of T is in an m-cycle for all $m \in \{3,4,\dots,n\}$, and Alspach [1] proved that every arc of a regular tournament on $n \geqslant 3$ vertices is in a k-cycle for all $k \in \{3,4,\dots,n\}$.

The structure of cycles in semicomplete multipartite digraphs (in particular, in multipartite tournaments) has been well studied (see [6]). Examples of Bondy [2] show that Moon's theorem is not valid for strong multipartite tournaments in general. However, from different extensions of the notion of a cycle in a tournament, one obtains various generalizations of Moon's theorem to multipartite tournaments (see [3,4]).

Goddard and Oellermann [3] proved that every vertex of a strongly connected n-partite ($n \geqslant 3$) tournament is in a cycle that contains vertices from exactly m partite sets for all m with $3 \leqslant m \leqslant n$. Volkmann and the present author [4] showed that if D is a strong n-partite ($n \geqslant 3$) tournament, then every partite set of D has a vertex v such that v is in an m-cycle for each m satisfying $3 \leqslant m \leqslant n$.

Since every strongly connected semicomplete n-partite digraph D with $n \geqslant 3$ contains a strongly connected n-partite tournament D' with $V(D') = V(D)$ (see Theorem 2.2 in [8]), the aforementioned two results are also valid for strongly connected semicomplete multipartite digraphs.

On the other hand, the result of Alspach above is not valid for regular n-partite tournaments. For example, the digraph D with $V(D) = \{a_1,a_2\} \cup \{b_1,b_2\} \cup \{c_1,c_2\}$ such that $a_1 \to \{b_1,c_1\} \to a_2 \to \{b_2,c_2\} \to a_1$ and $b_1 \to c_1 \to b_2 \to c_2 \to b_1$ is a regular 3-partite tournament, but the arc a_1b_1 is not in a 3-cycle.

In this paper, we introduce another extension of the notion of a cycle in tournaments. Observe that a vertex v of a tournament T is in a k-cycle if and only if T contains a $(k-1)$-path starting at v such that v does not dominate the endvertex of the path.

In general, we define an *outpath* of a vertex x (an arc xy, respectively) in a digraph as a path starting at x (xy, respectively) such that x dominates the endvertex of the path only if the endvertex also dominates x. An outpath of length k is called a k-outpath.

Note that an outpath of an arc xy in a multipartite tournament is a path starting from xy such that x does not dominate the endvertex of the path, and an arc of a tournament has a k-outpath if and only if the arc lies in a $(k+1)$-cycle. So, instead of cycles, we consider the outpaths in semicomplete multipartite digraphs. Firstly, we show that if D is a strongly connected, semicomplete n-partite ($n \geqslant 3$) digraph, then every vertex v of D has an outpath of length $k-1$ for all $k \in \{3,4,\dots,n\}$. Our result generalizes the aforementioned theorem of Moon [7] for tournaments. Secondly, we show that if T is a regular n-partite ($n \geqslant 3$) tournament, then every arc of T has an outpath of length $k-1$ for all k satisfying $3 \leqslant k \leqslant n$. It is immediate that this result is a generalization of Alspach's theorem [1] for regular tournaments.

2. Main results

Theorem 2.1. *Let D be a strongly connected semicomplete n-partite digraph with $n \geqslant 3$ and let v be a vertex of D. Then v has a $(k–1)$-outpath for each $k \in \{3, 4, \dots, n\}$.*

Proof. Since every strong semicomplete n-partite $(n \geqslant 3)$ digraph contains a spanning, strong n-partite tournament (see Theorem 2.2 in [8]), we prove the theorem under the assumption that D is a strong n-partite tournament.

Let $V_1, V_2, \dots, V_n$ be the partite sets of D and assume without loss of generality that $v \in V_1$. If $V_1 = \{v\}$, then, by the main result in [4] or [5], v is in a k-cycle, and hence, v has a $(k-1)$-outpath, for all $k \in \{3, 4, \dots, n\}$.

Suppose now that $|V_1| \geqslant 2$. By adding arcs from $V_1 \backslash \{v\}$ to v, we obtain an $(n+1)$-partite tournament D' that is also strong. Note that the vertex v forms a partite set by itself in D'. By the same argument as above, D' contains a k-cycle C_k through v for every $k \in \{3, 4, \dots, n, n+1\}$. If C_k contains an arc from $V_1 \backslash \{v\}$ to v, we delete it and obtain a $(k-1)$-outpath of v in D. $\square$

Theorem 2.2. *Let T be a regular n-partite tournament with $n \geqslant 3$. Then every arc of T has a $(p-1)$-outpath for all p satisfying $3 \leqslant p \leqslant n$.*

Proof. Let $V_1, V_2, \dots, V_n$ be the partite sets of T. From the regularity of T, it is not difficult to check that all partite sets of T have the same cardinality, say k. So, it is clear that

$$|N^+(x)| = |N^-(x)| = \frac{(n-1)k}{2} \quad \text{for each } x \in V(T).$$

Let a_1a_2 be an arbitrary arc of T and assume without loss of generality that $a_i \in V_i$ for $i = 1, 2$.

Because of $|N^+(a_1)| = |N^+(a_2)| = (n-1)k/2$ and $a_2 \in N^+(a_1)$, a_2 has at least one out-neighbor in $N^-(a_1) \cup (V_1 - a_1)$, and hence a_1a_2 has a 2-outpath.

To prove that a_1a_2 also has a 3-outpath when $n \geqslant 4$, we only need to show that one out-neighbor of a_2 has an out-neighbor in $N^-(a_1) \cup (V_1 - a_1)$. If a_2 has an out-neighbor a in $N^+(a_1)$, then a has at least one out-neighbor in $N^-(a_1) \cup (V_1 - a_1)$ by the regularity of T; if a_2 has no out-neighbor in $N^+(a_1)$, but it has an out-neighbor a' in $V_1 - a_1$, then $|N^+(a') \cap N^+(a_1)| \leqslant (n-1)k/2 - 1$, and hence, a' has at least one out-neighbor in $N^-(a_1)$; in the remaining case when a_2 has no out-neighbor in $N^+(a_1) \cup (V_1 - a_1)$, we have $a_2 \to N^-(a_1)$ by the regularity of T again. Because of $n \geqslant 4$, the subdigraph induced by $N^-(a_1)$ contains at least one arc, and we are done.

Thus, we have proved that every arc of T has a 2-outpath and a 3-outpath.

Suppose that a_1a_2 has a $(p-1)$-outpath, say $P = a_1a_2 \dots a_p$, but a_1a_2 has no p-outpath for some p with $4 \leqslant p < n$. Let

$$A = \{x \mid x \in V_i,\ V_i \cap V(P) = \emptyset,\ x \to a_1,\ 1 \leqslant i \leqslant n\},$$

and

$$B=\{y \mid y \in V_i,\ V_i \cap V(P)=\emptyset,\ a_1 \rightarrow y,\ 1 \leqslant i \leqslant n\}.$$

It is clear that $A \cup B \neq \emptyset$ and every vertex of $A \cup B$ is adjacent with all vertices in P. We consider the following two cases.

Case 1: $A \neq \emptyset$.

From the initial hypothesis that a_1a_2 has no p-outpath, we have that $A \rightarrow a_p$, and consequently, it is easy to check that $A \rightarrow P$. Let a be a vertex of A. From the proof above, the arc aa_{p-2} has a 2-outpath, say $aa_{p-2}a'$. Note that $a' \notin V(P)$. If a and a' are adjacent, then $a' \rightarrow a$, and hence, $a_1a_2 \ldots a_{p-2}a'aa_p$ is a p-outpath of a_1a_2, a contradiction. So, a and a' are in the same partite set of T. Since T is regular and $N^+(a) \cap N^-(a') \neq \emptyset$, $N^+(a') \cap N^-(a)$ contains at least one vertex, say u. Because of $A \rightarrow P$, u does not belong to P, and hence, $a_1a_2 \ldots a_{p-2}a'ua$ is a p-outpath of a_1a_2, a contradiction.

Case 2: $A=\emptyset$.

Let b be an arbitrary vertex of B, and assume without loss of generality that $b \in V_n$. Note that $V_n \subseteq B$.

Suppose that $a_i \rightarrow b$ for $i=2$ or $i=3$. Then, it is easy to check that $a_j \rightarrow b$ for all $j \geqslant 3$. Since the arc a_1b has a 2-outpath, there is a vertex x with $b \rightarrow x$ and $a_1 \nrightarrow x$. Obviously, $x \notin V(P)$, and hence, $a_1a_2 \ldots a_{p-1}bx$ is a p-outpath of a_1a_2, a contradiction. Therefore, $b \rightarrow \{a_2, a_3\}$. Thus, we have proved that $B \rightarrow \{a_2, a_3\}$.

Suppose now that $b \rightarrow a_p$. Then, it is easy to check that $b \rightarrow a_i$ for all $i \geqslant 2$. Because of $B \rightarrow a_2$ (in particular, $V_n \rightarrow a_2$), b is adjacent with all vertices of $N^+(a_2)$. Since $|N^+(b) \cap V(P)|=p-1$ and $|N^+(a_2) \cap V(P)| \leqslant p-2$, we deduce from the regularity of T that there is a vertex x with $a_2 \rightarrow x \rightarrow b$. Clearly, $x \notin V(P) \cup B$. Thus, $a_1a_2xba_4 \ldots a_p$ is a p-outpath of a_1a_2, a contradiction. Therefore, $a_p \rightarrow b$. From the choice of the vertex b, we have $a_p \rightarrow B$.

Let $p=4$. Since $V_n \subseteq B$ and $B \rightarrow a_2$, b is adjacent with all vertices of $N^+(a_2)$. From $b \rightarrow a_2$ and $|N^+(b)|=|N^+(a_2)|$, we conclude that there is a vertex u with $u \in N^+(a_2) \cap N^-(b)$. Clearly, $u \notin \{a_1, a_2, a_3\}$. Since a_1b has a 2-outpath, there exists a vertex w with $b \rightarrow w$ and $a_1 \nrightarrow w$. Obviously, $w \notin \{a_2, u\}$, and hence, a_1a_2ubw is a 4-outpath of a_1a_2, a contradiction.

In what follows, we consider the case when $p \geqslant 5$.

We first show that $a_{p-1} \rightarrow B$. Assume, on the contrary, that $b \rightarrow a_{p-1}$. Then $b \rightarrow a_i$ for each i satisfying $2 \leqslant i \leqslant p-1$. If there is a vertex $u \in N^+(a_2) \setminus V(P)$ with $u \rightarrow b$, then $a_1a_2uba_4 \ldots a_p$ is a p-outpath of a_1a_2, a contradiction. Hence, we have $b \rightarrow N^+(a_2) \setminus V(P)$. Recalling that $V_n \rightarrow a_2$ and $|N^+(a_2)|=|N^+(b)|$, we see from $|N^+(b) \cap V(P)|=p-2$ that $a_2 \rightarrow a_i$ for all $i \in \{3, 4, \ldots, p\}$. It follows that $a_1 \rightarrow \{a_3, a_4, \ldots, a_{p-1}\}$ (otherwise, $a_1a_2a_{\ell+1} \ldots a_pba_3 \ldots a_\ell$ is a p-outpath of a_1a_2 for some $\ell \in \{3, 4, \ldots, p-1\}$ with $a_1 \nrightarrow a_\ell$), and furthermore, $N^-(a_1) \setminus V(P) \rightarrow b$. But now, $|N^-(b)| \geqslant |N^-(a_1)|+|\{a_1\}|$, a contradiction. Therefore, $a_{p-1} \rightarrow B$.

To prove $a_4 \rightarrow B$ when $p \geqslant 6$, we assume that $b \rightarrow a_4$. It is a simple matter to verify that $(N^+(a_2) \setminus V(P)) \cap N^-(b)=\emptyset$. So, in the case $N^+(a_2) \setminus V(P) \neq \emptyset$, we have

$b \to N^+(a_2)\backslash V(P)$, and hence, we deduce from $a_{p-1} \to b$ that $a_1 \to N^+(a_2)\backslash V(P)$. If there is a vertex a_j with $a_2 \to a_j$, but $a_1 \nrightarrow a_{j-1}$, then $j \geqslant 4$ and $a_1a_2a_j \dots a_pba_3 \dots a_{j-1}$ is a p-outpath of a_1a_2, a contradiction. This implies that $|N^+(a_1) \cap V(P)| \geqslant |N^+(a_2) \cap V(P)|$. But now $|N^+(a_1)| \geqslant |N^+(a_2)| + |\{b\}|$, a contradiction.

From the proof above, we have $\{a_4, a_5, \dots a_p\} \to B \to \{a_2, a_3\}$. Since $a_{p-1} \to b$ and $a_1 \to V_n$, we have that $N^-(a_1)\backslash V(P) \to b$ if $N^-(a_1)\backslash V(P) \neq \emptyset$. Because of $|N^-(a_1)| = |N^-(b)|$ and $|N^-(b) \cap V(P)| = p-2$, we have $a_i \to a_1$ for all $i \geqslant 3$. It follows that $N^+(a_2) \cap V(P) = \{a_3\}$. Let w be a vertex of $N^+(a_2)\backslash\{a_3\}$. Since b is adjacent with all vertices of $N^+(a_2)$ and $|N^+(a_2)| = |N^+(b)|$, we see that $N^+(a_2) \cap N^-(b)$ contains at least one vertex, say x. But now, $a_1a_2xba_3a_4 \dots a_{p-1}$ is a p-outpath of a_1a_2, a contradiction. $\square$

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